ON THE STRUCTURE OF FINITE CONTINUOUS GROUPS WITH ONE TWO-PARAMETER INVARIANT SUBGROUP.

BY S. D. ZELDIN.

In a paper published in these Annals* I have considered groups having exceptional transformations and have shown how their structure can be simplified by imposing certain conditions on groups isomorphic with the given ones. In the present paper I shall show how the structure of groups with *one two-parameter invariant subgroup* can be simplified by imposing a few conditions on the groups *meroëdrically isomorphic* with them.

1. **Introductory remarks and assumptions.** Let G_{r+2} be a finite continuous group of order $r + 2$ generated by the infinitesimal transformations whose differential operators are $X_1, \cdots, X_r, X_{r+1}, X_{r+2}$, where

$$X_i = \sum_{k=1}^{r+2} \xi(x_1, \cdots, x_{r+2}) \frac{\partial}{\partial x_k} \qquad (i = 1, \cdots, r + 2),$$

and let G_{r+2} have an invariant two-parameter subgroup, which for simplicity may be taken to be generated by the operators X_{r+1} and X_{r+2}. Denoting the operators of the adjoint of G_{r+2} by $E_1, \cdots, E_{r+2}$, where

$$E_i = \sum_{j=1}^{r+2} \sum_{k=1}^{r+2} \alpha_j c_{jik} \frac{\partial}{\partial \alpha_k} \qquad (i = 1, \cdots, r + 2)$$

(the α's are the parameters of G_{r+2} and the c_{jik}'s are the structural constants), we can write down the following known equalities:

$$(1) \qquad (X_i, X_j) = \sum_{k=1}^{r} c_{ijk} X_k + \sum_{k=r+1}^{r} c_{ijk} X_k \quad (i, j = 1, \cdots, r),$$

$$(2) \qquad (X_i, X_j) = \sum_{k=r+1}^{r+2} c_{ijk} X_k \quad \begin{pmatrix} i = 1, \cdots, r+1, r+2 \\ j = \qquad\qquad r+1, r+2 \end{pmatrix},$$

$$(3) \qquad (E_i, E_j) = \sum_{k=1}^{r+2} c_{ijk} E_k \qquad (i, j = 1, \cdots, r).$$

Since the group G_{r+2} is assumed to have an invariant subgroup of order 2, there exists a *simple* group of order r, say G_r, which is meroëdrically isomorphic with G_{r+2}.† If we denote the operators of G_r by $Y_1, \cdots, Y_r$,

* Vol. 22, p. 95.

† Lie-Engel, vol. 3, p. 703.

where

$$Y_i = \sum_{k=1}^{r} \theta_{ki}(y_1, \cdots, y_r) \frac{\partial}{\partial y_k},$$

and the operators of the adjoint of G_r by $A_1, \cdots, A_r$, where

$$A_i = \sum_{k=1}^{r} \sum_{j=1}^{r} \alpha_j c_{jik} \frac{\partial}{\partial \alpha_k},$$

we have

$$(Y_i, Y_j) = \sum_{k=1}^{r} c_{ijk} Y_k \qquad (i, j = 1, \cdots, r)$$

and

$$(A_i, A_j) = \sum_{k=1}^{r} c_{ijk} A_k \qquad (i, j = 1, \cdots, r).$$

The condition imposed on the adjoint of G_r is that it shall have *one invariant spread.* It is to be observed that this spread is not a flat, for if it were, the group G_r would not be simple. If the invariant spread is given by the equation

$$F(\alpha_1, \cdots, \alpha_r) = 0,$$

then the function $F(\alpha_1, \cdots, \alpha_r)$ will satisfy the system of partial differential equations

$$A_i f(\alpha_1, \cdots, \alpha_r) \equiv \Sigma \alpha_j c_{ji1} \frac{\partial f}{\partial \alpha_1} + \cdots + \Sigma \alpha_j c_{jir} \frac{\partial f}{\partial \alpha_r} = 0 \quad (i = 1, \cdots, r).$$

Forming the matrix of the coefficients of those differential equations

$$\sum_{j=1}^{r} \alpha_j A_j \equiv \begin{vmatrix} \Sigma \alpha_j c_{j11}, & \Sigma \alpha_j c_{j12}, & \cdots & \Sigma \alpha_j c_{j1r} \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot \\ \Sigma \alpha_j c_{jr1}, & \Sigma \alpha_j c_{jr2}, & \cdots & \Sigma \alpha_j c_{jrr} \end{vmatrix}$$

we must have, since that system of equations has only one solution, the nullity of this matrix to be equal to *one*, i.e., at least one minor of order $r - 2$ of the determinant $|\Sigma \alpha_j A_j|$ does not vanish. But each minor of $|\Sigma \alpha_j A_j|$ is also a minor of $|\Sigma \alpha_j E_j|$, where

$$\sum_{j=1}^{r+2} \alpha_j E_j \equiv \begin{vmatrix} \Sigma \alpha_j c_{j,1,1}, & \cdots & \Sigma \alpha_j c_{j,r,1}, & \Sigma \alpha_j c_{j,r+1,1}, & \Sigma \alpha_j c_{j,r+2,1} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \Sigma \alpha_j c_{j,1,r}, & \cdots & \Sigma \alpha_j c_{j,r,r}, & \Sigma \alpha_j c_{j,r+1,r}, & \Sigma \alpha_j c_{j,r+2,r} \\ \Sigma \alpha_j c_{j,1,r+1}, & \cdots & \Sigma \alpha_j c_{j,r,r+1}, & \Sigma \alpha_j c_{j,r+1,r+1}, & \Sigma \alpha_j c_{j,r+2,r+1} \\ \Sigma \alpha_j c_{j,1,r+2}, & \cdots & \Sigma \alpha_j c_{j,r,r+2}, & \Sigma \alpha_j c_{j,r+1,r+2}, & \Sigma \alpha_j c_{j,r+2,r+2} \end{vmatrix}.$$

Therefore at least one minor of order $r - 3$ of the determinant $|\Sigma\alpha_j E_j|$ does not vanish, and thus the nullity of the matrix $\Sigma\alpha_j E_j$ can not exceed 3 for an arbitrary system of values of the α's. Further, for $\alpha_1, \cdots \alpha_{r+2}$ assigned, the symbolic equation

$$(A) \qquad \left(\sum_{i=1}^{r+2}\alpha_i X_i, \sum_{j=1}^{r+2}\beta_j X_j\right) = \rho\Sigma\Sigma\alpha_i\beta_j\sum_{k=r+1}^{r+2}c_{ijk}X_k \qquad (\rho \neq 0)$$

is satisfied by the following three independent solutions:

$$\begin{array}{llllllr} \beta_1 = \alpha_1, & \beta_2 = \alpha_2, & \cdots, & \beta_r = \alpha_r, & \beta_{r+1} = 0, & \beta_{r+2} = 0, & (1)\\ \beta_1 = 0, & \beta_2 = 0, & \cdots, & \beta_r = 0, & \beta_{r+1} = 1, & \beta_{r+2} = 0, & (2)\\ \beta_1 = 0, & \beta_2 = 0, & \cdots, & \beta_r = 0, & \beta_{r+1} = 0, & \beta_{r+2} = 1. & (3) \end{array}$$

The first set of β's satisfies, because it makes both sides of equation (A) equal to zero; the sets (2) and (3) satisfy because X_{r+1} and X_{r+2} form, by our assumption, an invariant subgroup of G_{r+2}.*

But from equation (A) follows the system of equations

$$\beta_1\sum_{i=1}^{r+2}\alpha_i c_{i11} + \beta_2\sum_{i=1}^{r+2}\alpha_i c_{i21} + \cdots + \beta_{r+1}\sum_{i=1}^{r+2}\alpha_i c_{i,\, r+1,\, 1} + \beta_{r+2}\sum_{i=1}^{r+2}\alpha_i c_{i,\, r+2,\, 1} = 0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\beta_1\sum_{i=1}^{r+2}\alpha_i c_{i2r} + \beta_2\sum_{i=1}^{r+2}\alpha_i c_{i2r} + \cdots + \beta_{r+1}\sum_{i=1}^{r+2}\alpha_i c_{i,\, r+1,\, r} + \beta_{r+2}\sum_{i=1}^{r+2}\alpha_i c_{i,\, r+2,\, r} = 0,$$

$$(1 - \rho)\Big[\beta_1\sum_{i=1}^{r+2}\alpha_i c_{i,\,1,\,r+1} + \beta_2\sum_{i=1}^{r+2}\alpha_i c_{i,\,2,\,r+1} + \cdots + \beta_{r+1}\sum_{i=1}^{r+2}\alpha_i c_{i,\,r+1,\,r+1} + \beta_{r+2}\sum_{i=1}^{r+2}\alpha_i c_{i,\,r+2,\,r+2}\Big] = 0,$$

$$(1 - \rho)\Big[\beta_1\sum_{i=1}^{r+2}\alpha_i c_{i,\,1,\,r+2} + \beta_2\sum_{i=1}^{r+2}\alpha_i c_{i,\,2,\,r+2} + \cdots + \beta_{r+1}\sum\alpha_i c_{i,\,r+1,\,r+2} + \beta_{r+2}\sum_{i=1}^{r+2}\alpha_i c_{i,\,r+2,\,r+2}\Big] = 0.$$

The determinant of the coefficients of the β's in those equations must have at least one non-vanishing minor of order $r - 1$, and therefore the nullity of the matrix $\sum_{j=1}^{r+2}\alpha_j E_j$ can not be less than 3. We may now say that the nullity of $\sum_{j=1}^{r+2}\alpha_j E_j$ is *equal to 3*. Now, since the nullity of the matrix $\Sigma\alpha_j E_j$ is equal to the number of independent invariants of the adjoint of G_{r+2}, the system of partial differential equations

$$E_1 f(\alpha) = 0, \qquad \cdots, \qquad E_{r+2} f(\alpha) = 0$$

* Since all the operators of the group G_{r+2} are given, ρ and c_{ijk} $(k = r + 1, r + 2)$ can easily be found.

has three independent functions in $\alpha_1, \cdots, \alpha_{r+2}$ for solutions. It is also evident that $F(\alpha_1, \cdots, \alpha_r)$, which is a solution of the equations

$$A_1 f(\alpha) = 0, \qquad \cdots, \qquad A_r f(\alpha) = 0,$$

is also a solution of

$$E_1 f(\alpha) = 0, \qquad \cdots, \qquad E_{r+2} f(\alpha) = 0,$$

for $F(\alpha)$ does not depend on $\alpha_{r+1}, \alpha_{r+2}$. Denoting the invariants of the adjoint of G_{r+2} by $F(\alpha_1, \cdots, \alpha_r)$, $V(\alpha_1, \cdots, \alpha_{r+2})$, $W(\alpha_1, \cdots, \alpha_{r+2})$, we may state the following

THEOREM. *If the adjoint of G_r, which is meroëdrically isomorphic with G_{r+2}, has one invariant, the adjoint of G_{r+2} has three invariants, one of which is also invariant to the adjoint of G_r.*

2. **The invariant spreads of the adjoint of G_{r+2} and their properties.** Consider the invariant spread $V(\alpha_1, \cdots, \alpha_{r+2}) = 0$, supposing that it is the only $(r + 1)$-flat invariant to the adjoint of G_{r+2}. It will then represent an invariant subgroup of order $r + 1$ of G_{r+2}.* It is to be observed that the two-parameter subgroup X_{r+1}, X_{r+2} which was assumed to be invariant in G_{r+2} represents geometrically a straight-line invariant in the space of the adjoint of G_{r+2}. We shall denote, in what follows, that line by the symbol $X_{r+1} \leftarrow\rightarrow X_{r+2}$. Now, if the invariant flat $V(\alpha_1, \cdots, \alpha_{r+2}) = 0$ does not pass through the line $X_{r+1} \leftarrow\rightarrow X_{r+2}$, there will then be in G_{r+2} an invariant subgroup of order $r + 1$ in addition to the given two-parameter invariant subgroup. In other words, we can find a new set of operators $\overline{X}_1, \cdots, \overline{X}_{r+1}, \overline{X}_{r+2}$, such linear functions of the old X's, that

$$(\overline{X}_i, \overline{X}_j) = \sum_{k=1}^{r+1} c_{ijk} X_k \quad \begin{pmatrix} i = 1, \cdots, r+1, r+2 \\ j = 1, \cdots, r+1 \end{pmatrix}.$$

If however the flat $V(\alpha_1, \cdots, \alpha_{r+2}) = 0$ does pass through the line $X_{r+1} \leftarrow\rightarrow X_{r+2}$, then the point of intersection would have to be invariant to adjoint of G_{r+2} and there would be in G_{r+2} an invariant subgroup of order one. This case brings us to exceptional transformations which I have already discussed in my last paper.

Suppose now that $V(\alpha_1, \cdots, \alpha_{r+2}) = 0$ is an equation of degree two, reducible to two linear equations, say $V_1(\alpha_1, \cdots, \alpha_{r+2}) = 0$ and $V_2(\alpha_1, \cdots, \alpha_{r+2}) = 0$, then the intersection of these two $(r + 1)$-flats will give an invariant r-flat in the space of the adjoint of G_{r+2}. If this r-flat does not pass through the line $X_{r+1} \leftarrow\rightarrow X_{r+2}$, we will be able to find r independent operators forming an invariant subgroup of order r of G_{r+2}.

* Lie-Scheffers, p. 479.

An interesting case will arise when $V(\alpha_1, \cdots, \alpha_{r+2}) = 0$ is an irreducible algebraic spread of degree $m \geqq 2$. Let us consider an arbitrary point P on the line $X_{r+1} \leftarrow\rightarrow X_{r+2}$. Its polar $(r+1)$-flat, with respect to the spread $V(\alpha_1, \cdots, \alpha_{r+2}) = 0$, will in general pass through some other point, say Q, of the line $X_{r+1} \leftarrow\rightarrow X_{r+2}$, and the polar $(r+1)$-flat of Q, with respect to the same spread, will then pass through P. The intersection of those two $(r+1)$-flats will give an r-flat which may be regarded as a polar r-flat, with respect to $V(\alpha_1, \cdots, \alpha_{r+2}) = 0$, of the line PQ (or $X_{r+1} \leftarrow\rightarrow X_{r+2}$). That flat may be looked upon as the locus of the poles of all $(r+1)$-flats passing through the line $X_{r+1} \leftarrow\rightarrow X_{r+2}$ taken with respect to the spread $V(\alpha_1, \cdots, \alpha_{r+2}) = 0$.*

Now, since the line $X_{r+1} \leftarrow\rightarrow X_{r+2}$ and the spread $V(\alpha) = 0$ are invariant to the adjoint of G_{r+2}, the aggregate of flats passing through the line $X_{r+1} \leftarrow\rightarrow X_{r+2}$ and therefore the locus of their poles quâ $V(\alpha) = 0$ will be invariant. Since that locus of poles is an r-flat, it follows that the group G_{r+2} has *an invariant subgroup of order* r, i.e., by properly choosing the operators $\overline{X}_1, \cdots, \overline{X}_{r+2}$ we shall have

$$(\overline{X}_i, \overline{X}_j) = \sum_{k=1}^{r} \bar{c}_{ijk} X_k \quad \begin{pmatrix} i = 1, \cdots, r, r+1, r+2 \\ j = 1, \cdots, r \end{pmatrix}.$$

Suppose, however, that $V(\alpha) = 0$ is of degree m, but is reducible to m $(r+1)$-flats. Then their common intersection (if there is any) will form an invariant $(r-m+2)$-flat. If that flat does not pass through the line $X_{r+1} \leftarrow\rightarrow X_{r+2}$, then there will be in G_{r+2} an *invariant subgroup of order* $r - m + 2$ *in addition to the given two-parameter subgroup*, i.e., the operators

$$\overline{X}_1, \cdots, \overline{X}_{r-m+2}, \overline{X}_{r-m+1}, \cdots, \overline{X}_r, \overline{X}_{r+1}, \overline{X}_{r+2}$$

can be so chosen that

$$(\overline{X}_i, \overline{X}_j) = \sum_{k=1}^{r-m+2} \bar{c}_{ijk} X_k \qquad \begin{pmatrix} i = 1, \cdots, r-m+2, \cdots, r \\ j = 1, \cdots, r-m+2 \end{pmatrix}$$

$$(X_i, X_{r+1}) = (X_i, X_{r+2}) = 0 \qquad (i = 1, \cdots, r-m+2)$$

$$(X_i, X_j) = \sum_{r+1}^{r+2} \bar{c}_{ijk} X_k \qquad \begin{pmatrix} i = r-m+2, \cdots, r+1, r+2 \\ j = \qquad\qquad r+1, r+2 \end{pmatrix}.$$

If the $(r-m+2)$-flat does pass through the line $X_{r+1} \leftarrow\rightarrow X_{+2}$, we again get a single invariant point in the space of the adjoint of G_{r+2}, whose meaning I discussed before.

It may happen that $V(\alpha) = 0$ breaks up into spreads each of degree greater than one. Then their common intersection and its polar flat,

* Compare with Salmon's discussion of polar lines, G. Salmon, Analytic Geometry of Three Dimensions, p. 49.

taken with respect to the line $X_{r+1} \leftarrow\rightarrow X_{r+2}$, can be considered exactly in the same way as when $V(\alpha) = 0$ was irreducible.

So far I have only considered the invariant $V(\alpha)$ independently of the third invariant $W(\alpha)$ of the adjoint of G. We could of course obtain the same results for $W(\alpha)$, as we did for $V(\alpha)$, by considering it alone. Suppose, however, that the invariant spreads

$$V(\alpha) = 0 \qquad \text{and} \qquad W(\alpha) = 0$$

are taken together, and assume first that both are $(r + 1)$-flats. If their intersection, which is an r-flat, does not pass through the line $X_{r+1} \leftarrow\rightarrow X_{r+2}$, then there is an invariant subgroup of order r of G_{r+2} in addition to the given invariant two-parameter subgroup. If however the $(r + 1)$-flats do not intersect at all, then each one separately will represent an invariant subgroup of order $r + 1$ of G_{r+2}.

If finally $V(\alpha) = 0$ and $W(\alpha) = 0$ are spreads of degrees m and n respectively, then, by considering the polar flat of the line $X_{r+1} \leftarrow\rightarrow X_{r+2}$, taken with respect to the intersection of $V(\alpha) = 0$ and $W(\alpha) = 0$, we shall get an invariant subgroup of order r of G_{r+2}.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY,
November 8, 1920.